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ON A BIRATIONAL TRANSFORMATION CONNECTED WITH A PENCIL OF CUBICS

BY

ARTHUR ROBINSON WILLIAMS

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PREFATORY NOTE

When I prepared the following paper I was acquainted only with the paper by Cayley, cited on page 212, and the articles therein mentioned. The problems of the construction of the cubic through nine points and the construction of the ninth point of intersection of all cubics through eight points, virtually the same problem, have been treated by Weddle, by Hart, by Chasles, by Cayley, and of course by later writers. Weddle's paper may be found in the *Cambridge and Dublin Mathematical Journal,* Vol. VI (1851), page 83, and Hart's in the same volume, page 181. Chasles' memoir appeared in *Comptes rendus lebdomadaire des seances de l'Academie des Sciences,* Vol. XXXVI (1853), page 942.

It does not seem to have occurred to these writers to inquire concerning the locus of the ninth point when seven remain fixed and the eighth moves in a given manner. However, Professor H. S. White, to whom this paper was referred, pointed out that the subject had been well covered by Geiser and Milinowski.

The paper by Geiser, "*Ueber Zwei Geometrische Probleme,*" is in *Crelle's Journal,* Vol. 67 (1867), page 78. Starting from the well-known fact, which follows from the elementary properties of the cubic, that if seven points be fixed and an eighth be taken on the line joining two of them, the ninth intersection of all cubics through the eight lies on the conic through the other five, and conversely; and making use of a theorem of Steiner's regarding the locus of the double points of all the cubics through seven points, he derives by synthetic and very simple methods practically all the results which I have obtained, including the special cases arising from relations among the seven fixed points. Naturally neither he nor Milinowski, whose method is also synthetic, mentions the system of lines which I consider in section 12.

Milinowski, whose paper on this subject is found in *Crelle's Journal,* Vol. 77 (1874), page 263, sets up a one-to-one correspondence between the cubics through seven points and the points of the plane. The same end is served by making cor-

respond to a given cubic of the net its polar line with respect to a fixed point. To obtain the locus of the ninth point when seven are fixed and the eighth moves, I have expressed its co-ördinates directly in terms of those of the eighth. This would not be feasible of course for curves of higher order than the third. Thus in case of a net of quartics, if twelve points are fixed, there correspond to a thirteenth not one but three others. Milinowski's methods are equally applicable to this and to curve nets of higher order.

The second and longer portion of Geiser's paper is given to the discussion of an analogous problem in space. All quadric surfaces through seven fixed points have an eighth in common. Hesse, then *privatdocent* at Konigsberg, has given a construction for this eighth point in *Crelle's Journal*, Vol. 26 (1843), page 147. Geiser discusses the locus of the eighth point when six are fixed and the seventh moves in a given manner. The problem is rather more complex than that of the net of plane cubics, and the results obtained are very curious and interesting.

The theorem that all the cubics through eight points have a ninth in common, suggests the following transformation, which is evidently birational. Fix seven points, and make correspond to a variable eighth the remaining common point of all the cubics through the eight.

1. *Construction of the ninth intersection of the cubics through eight points.*

It follows from the properties of the cubic regarded as the locus of intersections of corresponding elements of a pencil of conics and a pencil of lines projectively related, that if any four of eight points be taken, the ninth point of intersection of all the cubics through the eight projects to the four selected points in four rays whose anharmonic ratio is the same as that of the four conics which have the four remaining points in common, and are determined respectively by the four chosen points.[1] Therefore 9 is seen to lie on a certain conic through the four chosen points. In particular, if we take two sets of four points which have three points in common, 9 will be determined as the fourth intersection of two conics three of whose intersections are known; or, in other words, as the point which projects simultaneously to two sets of four points (which have three points in common) in two sets of four rays, each four having a given anharmonic ratio. Thus $9\{4\,6,\,7\,8\}=(1\,2\,3\,5)\{4\,6,\,7\,8\}$, and $9\{5\,6,\,7\,8\}=(1\,2\,3\,4)\{5\,6,\,7\,8\}$; where $(1\,2\,3\,5)\{4\,6,\,7\,8\}$ means the anharmonic ratio of the four conics which have 1 2 3 5 in common, and which pass through 4 6 7 8 respectively. In these anharmonic ratios the order of elements is arbitrary, but once chosen must be consistently maintained. In the above notation, primary and secondary elements are separated by a comma. Cayley, in the paper cited, has given the following construction due to Dr. Hart for a ninth point so determined:

Let $(1\,2\,3\,5)\{4\,6,\,7\,8\}$ be denoted by p, and $(1\,2\,3\,4)\{5\,6,\,7\,8\}$ by s. The anharmonic ration of four conics of a pencil is given by that of their four points of inter-

[1] Cayley "On the Construction of the Ninth Point of Intersection of the Cubics which Pass through Eight Points." *Collected Papers*, Vol. 4, p. 495.

section with any line passing through any one of their four common points, or by that of the four tangents at one of the common points, or again by that of the po'ars of any point with respect to the four conics. After determining p and s, Dr. Hart proceeds thus: Let 74 and 65 meet in M, and on 74 take Q so that $\{4M, 7Q\} = p$. Let 85 meet 64 in N, and on 85 take R so that $\{5N, R8\} = s$. Let QR meet 65 in K and 64 in L. Then $7K$ and $8L$ give 9. The whole construction may be accomplished with the ruler.

2. *Analytic treatment of this construction shows that the co-ördinates of* 9, $\lambda'\mu'\nu'$, *are octavic functions of the co-ördinates of* 8, *of the form* $\lambda' = C_2C_3K_1$, $\mu' = C_3C_1K_2$, $\nu' = C_1C_2K_3$, *where the Cs are of degree three and the Ks of degree two.*

Now introducing coördinates, let 4 5 6 be the vertices of the triangle of reference; 6 being $(1 : 0 : 0)$, 5 being $(0 : 0 : 1)$, and 4 $(0 : 1 : 0)$. Then 4 5 is the line $x=0$, 56 is $y=0$, and 64 is $z=0$. Let 1 2 3 7 be $(\lambda_1 : \mu_1 : \nu_1)$, $(\lambda_2 : \mu_2 : \nu_2)$, etc., and let 8, which is to be regarded as variable, be $(\lambda : \mu : \nu)$. The quantities p and s are now themselves functions of $\lambda\ \mu\ \nu$. p is $(1\,2\,3\,5)\{4\,6,\ 7\,8\}$. The four conics all pass through 5, and it will therefore suffice to find the anharmonic ration of their four other points of intersection with the line 4 5, i.e., the line $x=0$. The conic $1\,2\,3\,5-4$ meets $x=0$ at 4, i.e. $(0 : 1 : 0)$. The equation of $1\,2\,3\,5-6$ may be written thus:

$$\begin{vmatrix} y^2 & xy & xz & yz \\ \mu_1^2 & \lambda_1\mu_1 & \lambda_1\nu_1 & \mu_1\nu_1 \\ \mu_2^2 & \lambda_2\mu_2 & \lambda_2\nu_2 & \mu_2\nu_2 \\ \mu_3^2 & \lambda_3\mu_3 & \lambda_3\nu_3 & \mu_3\nu_3 \end{vmatrix} = 0$$

Setting $x=0$, we have $y : z = a : b$, where a and b are the minors of yz and y^2 respectively. Similarly the equation of $1235-7$ may be written in the determinant form, and, setting $x=0$, we have $y : z = f : g$ where $f \equiv (\lambda_1^2 \cdot \mu_2^2 \cdot \lambda_3\mu_3 \cdot \lambda_7\nu_7)$ and $g \equiv (\lambda_1^2 \cdot \lambda_2\mu_2 \cdot \lambda_3\nu_3 \cdot \mu_7\nu_7)$. For $1\,2\,3\,5-8$ we have the same equation except that the coördinates of 7 are replaced by those of 8. Setting $x=0$, we have $y : z = F_8 : G_8$, where F_8 and G_8 are obtained on replacing $\lambda_7\mu_7\nu_7$ by $\lambda\mu\nu$ in f and g. The capital letters and the subscript will indicate the presence of the coordinates of the variable point. Hence the four points whose anharmonic ratio we desire are respectively $(0 : 1 : 0)$, $(0 : a : b)$, $(0 : f : g)$, $(0 : F_8 : G_8)$. Following the order of elements indicated above, we have the anharmonic ratio of the four conics $g\,(aG_8 - bF_8)/G_8(ag - bf) \equiv p$. In the same way the anharmonic ratio $(1\,2\,3\,4)\ \{5\,6,\ 7\,8\}$ is found to be $g(bH_8 - cG_8)/G_8(bh - cg)$. Only three new determinants have been introduced, $c \equiv (\nu_1^2 \cdot \lambda_2\mu_2 \cdot \lambda_3\nu_3)$, $h \equiv \lambda_1^2 \cdot \nu_2^2 \cdot \lambda_3\mu_3 \cdot \lambda_7\nu_7)$, and H_8, which is obtained on replacing $\lambda_7\mu_7\nu_7$ by $\lambda\mu\nu$ in h. Thus a, b, c are of the third order and involve only the coördinates of 123. The other determinants are of the fourth order and related as indicated.

In accordance with the construction outlined, we take Q on 7 4 so that $\{4M, 7Q\} = p$, where M is the intersection of 74 with 65, that is, of 7 4 with $y=0$. The

coördinates of Q are thus found to be ;$(\lambda_7 : p\mu_7 : \nu_7)$. Similarly the intersection of 8 5 with 6 4 is N, and taking R on 8 5 so that $\{5N, R8\} = s$, R is $(s\lambda : \mu s : \nu)$. The intersection of QR with $y=0$ gives K, and with $z=0$ gives L. The intersection of $7K$ and $8L$ is 9. Thus for $\lambda'\mu'\nu'$, the coördinates of 9, we obtain

$$\left.\begin{aligned} \lambda' &= (\nu_7\lambda s - \lambda_7\nu)(\lambda_7\mu - \mu_7\lambda p)(p-s) \\ \mu' &= (\lambda_7\mu - \mu_7\lambda p)(\mu_7 p\nu - \nu_7\mu s)(s-1) \\ \nu' &= (\mu_7 p\nu - \nu_7\mu s)(\nu_7\lambda s - \lambda_7\nu)(1-p) \end{aligned}\right\} (1)$$

The following identities are at once evident

$$\left.\begin{aligned} &\lambda(\mu_7 p\nu - \nu_7\mu s) + \mu(\nu_7\lambda s - \lambda_7\nu) + \nu(\lambda_7\mu - \mu_7\lambda p) = 0 \\ &(p-s) + (s-1) + (1-p) = 0 \end{aligned}\right\} (\alpha)$$

To obtain $\lambda'\mu'\nu'$ as homogeneous expressions in $\lambda\mu\nu$ it is necessary only to substitute the values of p and s in (1). After removal of a common denominator and the factor g from each numerator, we have

$$\left.\begin{aligned} \lambda' = &\{\nu_7\lambda(bH_8 - cG_8)g - \lambda_7\nu(bh-cg)G_8\} \times \\ &\{\lambda_7\mu(ag-bf)G_8 - \mu_7\lambda(aG_8 - bF_8)g\} \times \\ &\{(bh-cg)(aG_8 - bF_8) - (ag-bf)(bH_8 - cG_8)\}. \\ \mu' = &\{\lambda_7\mu(ag-bf)G_8 - \mu_7\lambda(aG_8 - bF_8)g\} \times \\ &\{\mu_7\nu(bh-cg)(aG_8 - bF_8) - \nu_7\mu(ag-bf)(bH_8 - cG\ )\} \times \\ &\{(bH_8 - cG_8)g - (bh-cg)G_8\}. \\ \nu' = &\{\mu_7\nu(bh-cg)(aG_8 - bF_8) - \nu_7\mu(ag-bf)(bH_8 - cG_8)\} \times \\ &\{\nu_7\lambda(bH_8 - cG_8)g - \lambda_7\nu(bh-cg)G_8\} \times \\ &\{(ag-bf)G_8 - (aG_8 - bF_8)g\}. \end{aligned}\right\} (2)$$

Recalling that $F_8\ G_8\ H_8$ are determinants homogeneous and of degree two in $\lambda\ \mu\ \nu$, it is evident that $\lambda'\ \mu'\ \nu'$ are homogeneous of the eighth degree, and of the form

$$\left.\begin{aligned} \lambda' &= C_2C_3K_1 \\ \mu' &= C_3C_1K_2 \\ \nu' &= C_1C_2K_3 \end{aligned}\right\} (2')$$

where the Cs are of the third degree and the Ks of the second. The identities connecting the Cs and Ks are

$$\left.\begin{aligned} &\lambda g C_1 + \mu(ag-bf)C_2 + \nu(bh-cg)C_3 = 0 \\ &gK_1 + (ag-bf)K_2 + (bh-cg)K_3 = 0 \end{aligned}\right\} (\beta)$$

3. *The factors C and K may be replaced by determinants proportional to them.*

From the form of $F_8\ G_8\ H_8$ it follows that they all vanish when the coördinates, $\lambda\ \mu\ \nu$, of 8 are replaced by those of any one of the points 1 2 3. Then, expanding in terms of $\lambda\ \mu\ \nu$, it is easily seen that $G_8 = 0$, $aG_8 - bF_8 = 0$, $bH_8 - cG_8 = 0$ represent respectively the conics 1 2 3 4 5, 1 2 3 5 6, 1 2 3 4 6. Noting that the left-hand mem-

bers of these conics become g, $ag-bf$, $bh-cg$ when $\lambda_7\,\mu_7\,\nu_7$ are put for $\lambda\,\mu\,\nu$, it appears immediately that $C_2=0$, $C_3=0$, $C_1=0$ represent three nodal cubics, each of which passes through all the seven fixed points, and which have their double points at 4 5 6 respectively. Similarly $K_1=0$, $K_2=0$, $K_3=0$ represent three conics which have the points 1 2 3 7 in common and which pass through 6 4 5 respectively. The symmetry of $\lambda'\ \mu'\ \nu'$ is quite striking. Thus the two cubics that form part of λ' have their double points at (xy) and (xz), while the accompanying conic, K_1, passes through 6, i.e., the vertex opposite to $x=0$. Similarly for the factors of μ' and ν'.

Thus the form of each factor C and of each factor K is determined down to a constant multiplier by the character of the curve that it represents when equated to zero. For example $K_2=0$ gives the conic 1 2 3 7 4, and therefore K_2 differs at most by a constant factor from the determinant

$$\begin{vmatrix} \nu^2 & \lambda^2 & \mu\nu & \nu\lambda & \lambda\mu \\ \nu_1^2 & \lambda_1^2 & \mu_1\lambda_1 & \nu_1\lambda_1 & \lambda_1\mu_1 \\ \nu_2^2 & \lambda_2^2 & \mu_2\nu_2 & \nu_2\lambda_2 & \lambda_2\mu_2 \\ \nu_3^2 & \lambda_3^2 & \mu_3\nu_3 & \nu_3\lambda_3 & \lambda_3\mu_3 \\ \nu_7^2 & \lambda_7^2 & \mu_7\nu_7 & \nu_7\lambda_7 & \lambda_7\mu_7 \end{vmatrix}$$

Expanding this determinant by the elements of the first row, and expanding also $K_2\equiv(bH_8-cG_8)g-(bh-cg)G_8$, i.e. $b(gH_8-hG_8)$, comparison of the coefficients shows that K_2 is identical with the above determinant multiplied by $-bJ$. J is the determinant $(\lambda_1^2\cdot\lambda_2\mu_2\cdot\lambda_3\nu_3)\equiv\lambda_1\lambda_2\lambda_3(\lambda_1\cdot\mu_2\cdot\nu_3)$. Similar relations hold for the other Cs and Ks. Thus

$$\begin{aligned} K_1&\equiv(\mu^2\cdot\nu_1^2\cdot\mu_2\nu_2\cdot\nu_3\lambda_3\cdot\lambda_7\nu_7)J^2b \\ K_2&\equiv(\nu^2\cdot\lambda_1^2\cdot\mu_2\nu_2\cdot\nu_3\lambda_3\cdot\lambda_7\mu_7)(-Jb) \\ K_3&\equiv(\lambda^2\cdot\mu_1^2\cdot\mu_2\nu_2\cdot\nu_3\lambda_3\cdot\lambda_7\nu_7)Jb \\ C_2&\equiv(\lambda^2\mu\cdot\lambda_1^2\nu_1\cdot\lambda_2\mu_2\nu_2\cdot\lambda_3\nu_3^2\cdot\mu_7\nu_7^2)(-JJ_1) \\ C_3&\equiv(\lambda^2\mu\cdot\lambda_1^2\nu_1\cdot\lambda_2\mu_2^2\cdot\lambda_3\mu_3\nu_3\cdot\mu_7^2\nu_7)JJ_1 \\ C_1&\equiv(\lambda\mu^2\cdot\lambda_1\mu_1\nu_1\cdot\lambda_2\nu_2^2\cdot\mu_3^2\nu_3\cdot\mu_7\nu_7^2)J^2J_1 \end{aligned}$$

$J_1\equiv(\lambda_1\cdot\mu_2\cdot\nu_3)$. It is seen, therefore, that $\lambda'\ \mu'\ \nu'$ as given by (2) contain the common factor $-J_4J_1^2\equiv-\lambda_{14}\lambda_{24}\lambda_3 tJ_1$i.Removing this common factor, we may use for the Cs and Ks the determinants indicated in the last set of equations.

4. *Geometric character of the transformation.*

In (2) $\lambda'\ \mu'\ \nu'$ represent curves of the eighth order which have seven triple points in common, that is sixty-three $(=8^2-1)$ common intersections. Since the possession of a triple point at a given point imposes six conditions, it follows that a curve of order eight having seven fixed triple points is subjected to forty-two conditions, two less than the number sufficient to determine a curve of that order. This accords with the general theory of Cremona transformations which demands that in a birational transformation of degree n the expressions of the nth degree

shall represent curves which have n^2-1 common intersections, while these intersections must represent

$$\frac{n\,(n+3)}{2}-2$$

conditions, two less than the number required to determine a curve of order n.

Therefore if 8 moves on a line not passing through any of the seven fixed points, 9 will describe a proper, i.e., undegenerate, curve of order eight having a triple point at each of the seven fixed points. If 8 moves on a conic which does not pass through any of the seven fixed points, 9 will describe a curve of order sixteen having a sextuple point at each of them. According to the general theory there will correspond to each of the seven fixed points a curve of order three. The nature of this correspondence appears immediately from (2). For example, if 8 is any point on C_1, 9 is $(1 : 0 : 0)$. But C_1 is the cubic which has its double point at $(1 : 0 : 0)$. For 8 on C_3, 9 is $(0 : 0 : 1)$, and for 8 on C_2, 9 is $(0 : 1 : 0)$, which are the double points of C_3 and C_2 respectively. Considerations of symmetry enable us to conclude immediately that in this transformation the curve of degree three corresponding to any one of the seven fixed points is the nodal cubic passing through all of them and having a double point at the one in question. These seven cubics taken together constitute the Jacobian of $L\ M\ N$; where $L\ M\ N$ are the functions of $\lambda'\ \mu'\ \nu'$ got by solving (2) for λ, for μ, and for ν respectively. Since in this transformation the coördinates of 8 are evidently the same functions of the coördinates of 9 as the coördinates of 9 are of those of 8, the same seven cubics constitute the Jacobian of $\lambda'\ \mu'\ \nu'$. If 8 is on K_1 (2) shows that 9 is on $x=0$. But K_1 is the conic 12376 and $x=0$ is the line 4 5. Similarly for the conics K_2 and K_3 and the lines $y=0$ and $z=0$. It thus appears from (2), as is obvious from the general properties of the transformation and from those of the cubic, that to the line joining two of the fixed points corresponds the conic through the other five. Or, in other words, the curve of order eight corresponding to such a line consists of this conic and the two cubics which correspond to the two fixed points. With the exception of the seven special cubics just mentioned, all cubics through the seven fixed points are self-corresponding. For the points of any one of them correspond in pairs. To a sextic which has a double point at each of the original points corresponds another sextic which itself has a double point at each of them. It will be seen that there is one such sextic all of whose points are invariant. It is in fact the locus of the invariant points of the transformation.

5. *There are infinitely many invariant points, and their locus is a sextic.*

To ascertain the invariant points, we may put $\lambda\ \mu\ \nu$ for $\lambda'\ \mu'\ \nu'$ in (1). Then dividing the first equation by the second, the first by the third, and the second by the third, and cross-multiplying, we have the three equations, two of which are independent,

$$\lambda(\mu_7 p\nu-\nu_7\mu s)(s-1)-\mu(\nu_7\lambda s-\lambda_7\nu)(p-s)=0$$
$$\mu(\nu_7\lambda s-\lambda_7\nu)(1-p)-\nu(\lambda_7\mu-\mu_7\lambda p)(s-1)=0$$
$$\nu(\lambda_7\mu-\mu_7\lambda p)(p-s)-\lambda(\mu_7 p\nu-\nu_7\mu s)(1-p)=0$$

Any invariant points must be common to the three curves represented by these equations. But all three reduce to

$$\lambda ps(\mu_7\nu-\nu_7\mu)+\mu s(\nu_7\lambda-\lambda_7\nu)+\nu p(\lambda_7\mu-\mu_7\lambda)=0 \qquad (4)$$

Hence, instead of a finite number of invariant points there are infinitely many; in fact all the points of the curve (4). This curve is evidently a sextic, and, referring to (2′), may be written in any of the three forms

$$\begin{aligned}\lambda C_1K_2-\mu C_2K_1&=0,\\ \mu C_2K_3-\nu C_3K_2&=0,\\ \nu C_3K_1-\lambda C_1K_3&=0.\end{aligned}$$

Any of these equations shows that the sextic has a multiple point which is at least a double point at each of the seven fixed points. But no one of them can be of higher order than two, for in that case there would correspond to the sextic a curve of lower order than six.

An interesting property of the invariant points appears when they are considered in connection with the original construction. It will be recalled that 9 is the fourth intersection of two conics one of which is the locus of points that project to 4 6 7 8 in the anharmonic ratio p, while the points of the other project to 5 6 7 8 in the anharmonic ratio s. Let the tangents to these conics at 8 be l_p and l_s. Then the equations of these tangents may be easily obtained. For the anharmonic ratio of the four rays 84, 86, 87 and l_p is p, while that of 85, 95, 87, l_s is s. Making use of the fact that the anharmonic ratio of four lines of the form L, M, $L-rM$, $L-r_1M$ is r/r_1, and keeping the same order of elements as in calculating p and s, the equations of l_p and l_s are found to be respectively

$$\begin{aligned}p\nu x-r\nu y+(r\mu-\lambda p)z&=0\\ s\mu x-(k\nu+\lambda s)y+k\mu z&=0\end{aligned}$$

where $$r\equiv-\frac{\nu_7\lambda-\lambda_7\nu}{\mu_7\nu-\nu_7\mu}$$

and $$k\equiv\frac{\lambda_7\mu-\mu_7\lambda}{\mu_7\nu-\nu_7\mu}$$

The conics will have their fourth intersection at 8 if l_p and l_s coincide; that is if

$$\frac{p\nu}{s\mu}=\frac{r\nu}{k\nu+\lambda s}=\frac{r\mu-\lambda p}{k\mu},$$

the variables being λ μ ν, the coördinates of 8. Forming the three possible equations and bringing all the terms to the left hand member, we have in each case the

factor $\lambda ps-\mu sr+\nu pk$. Hence all points of the curve that correspond to this common factor are invariant. Replacing r and k by their values we have

$$\lambda ps(\mu_7\nu-\nu_7\mu)+\mu s(\nu_7\lambda-\lambda_7\nu)+\nu p(\lambda_7\mu-\mu_7\lambda)=0$$

which is (4) above. Substituting the values of p and s in (4), we obtain

$$\lambda(\mu_7\nu-\nu_7\mu)(aG_8-bF_8)(bH_8-cG_8)g+\mu(\nu_7\lambda-\lambda_7\nu)(ag-bf)(bH_8-cG_8)G_8 \\ +\nu(\lambda_7\mu-\mu_7\lambda)(bh-cg)(aG_8-bF_8)G_8=0. \tag{5}$$

It will be observed that at points 8 for which l_p and l_s coincide they have two intersections with all the cubics of the one parameter family through the seven fixed points and the point 8 in question. In other words the cubics of the family determined by the seven fixed points and an invariant point have a common tangent at the latter, whose equation is given equally by l_p and l_s. There is, however, one cubic of the family which has a double point at the invariant point, and for which the line l_{ps} is not in general a tangent. The envelope of this set of lines l_{ps} will be sought after a little further examination of the sextic.

In (5), note that $\mu_7\nu-\nu_7\mu$, $\nu_7\lambda-\lambda_7\nu$, $\lambda_7\mu-\mu_7\lambda$, when equated to zero are the lines 67, 47, 57 respectively; while $G_8=0$, $aG_8-bF_8=0$, $bH_8-cG_8=0$ are the conics 12345, 12356, 12346. Thus it appears that the sextic passes through the intersections of the conic 12345 with the line 67, of 12356 with 47, and of 12346 with 57. It is thus obvious from symmetry that the sextic passes through the intersections of the conic determined by any five of the seven fixed points with the line joining the other two. There are twenty-one such pairs making a total of forty-two points.

By a simple regrouping of the terms of (5) with change of signs, we obtain

$$\lambda G_8C_1+\mu(aG_8-bF_8)C_2+\nu(bH_8-cG_8)C_3=0 \tag{6}$$

where $C_1\,C_2\,C_3$ are defined as in (2). C_2 has a double point at 4, C_3 at 5, and C_1 at 6. Using (6), it is very easy to show by means of the second partial derivatives that at 4 the tangents to the sextic coincide with those of C_2, at 5 with those of C_3, and at 6 with those of C_1. It is necessary only to indicate the differentiation. To tell the terms that vanish, it suffices to remember through what points the conics pass and the double points of the cubics. Hence it appears immediately from symmetry that the sextic has a double point at each of the seven fixed points whose tangents coincide with those of the cubic that has the same double point and passes through the other six fixed points. Of the eighteen intersections of the sextic with any of the cubics of this system six are at their common double point and the remaining twelve at the other six points. The conic through any five of the fixed points has ten of its twelve intersections with the sextic at those points, and the other two are on the line determined by the two remaining points. The line joining any two of the fundamental points meets the sextic four times at those points and twice where it meets the conic through the other five. Thus the sextic fulfills seventy-seven conditions; twenty-one by reason of its seven fixed

double points, fourteen by reason of the determination of the tangents at those double points, and forty-two others corresponding to the twenty-one point pairs noted in the preceding paragraph.

One other property is easily obtained. $C_1C_2C_3$ are three linearly independent cubics through the seven fixed points. Hence any of the ∞^2 cubics through them is of the form $C \equiv \xi C_1 + \eta C_2 + \zeta C_3 = 0$ where ξ, η, ζ are constants. It follows immediately from the form of the first partial derivatives of C, that if C have a double point, it must lie on the Jacobian of $C_1C_2C_3$. But from the character of the latter it follows that their Jacobian is a sextic which must have a double point at each of the seven fixed points, and must have at 6, at 4, and at 5 the same tangents as C_1, C_2, C_3 respectively. Therefore it fulfills twenty-seven conditions which determine it uniquely as coincident with the invariant sextic.

Summarizing the results of this section we have the following theorem:

Given seven points in a plane, the (forty-two) points in which the line determined by any two meets the conic determined by the other five lie on a sextic which has a double point at each of the seven given points, and whose tangents at any one of these points coincide with those of the cubic that has the same double point and passes through the other six given points.

This sextic is the locus of the double points of all nodal cubics that pass through the seven given points.

It is also the locus of points 8 *such that the single infinity of cubics determined by* 8 *and the seven given points have a common tangent at* 8.

6. *The single infinity of cubics determined by the seven fixed points and an invariant point have a common tangent at the latter; and the envelope of the system of lines thus defined is of class four and order twelve.*

These common tangents are, as has been seen in section 5, paragraph 2, the lines l_p, or l_s, which coincide at points on the sextic. Their envelope, therefore, is the envelope of l_p, or, which is the same thing, that, of l_s, when 8, whose coördinates determine their coefficients, moves on the sextic. Difficulties of elimination render the more direct methods inapplicable. The class is, however, easily obtained. For if in l_p we fix xyz, the number of lines of the system through $(x:y:z)$ will depend on the intersections of the sextic with l_p regarded as a locus in $\lambda\mu\nu$. The latter is a quartic only four of whose intersections with the sextic depend on xyz. Hence the class is four. The same is obtained from l_s. It is not feasible, however, to express the condition that two of these four intersections coincide.

But the lines whose envelope we seek join 8 and 9, which for points on the sextic are infinitely near. From (1) the equation of the line connecting 8 with 9 is

$$x(\mu_7\nu p - \nu_7\mu s)\Sigma + y(\nu_7\lambda s - \lambda_7\nu)\Sigma + z(\lambda_7\mu - \mu_7\lambda p)\Sigma = 0$$

where Σ is the left hand member of the sextic given by (4). Removing this common factor the coefficients of xyz are easily shown to be proportional to the corres-

ponding coefficients in l_p and l_s for points on the sextic. Substituting the values of p and s in the equation of 8 9, we have

$$xgC_1+y(ag-bf)C_2+z(bh-cg)C_3=0 \tag{7}$$

which, for $\lambda\mu\nu$ variable, is a cubic through the seven fixed points, and through $(x:y:z)$ by the first of identities (β), end of section 2. It has therefore, as expected, four intersections with the sextic which depend on xyz, and at each of them it is tangent to the corresponding line of the system. Hence the four lines of the system through an arbitrary $(x':y':z')$ are the four tangents to the corresponding cubic (7) from $(x':y':z')$. If $(x':y':z')$ is an inflection, there are only three such tangents; but in that case $(x':y':z')$ is a point P on the sextic and the line of the system corresponding to P makes the fourth. For this line is the common tangent at P to all the cubics (7) determined by points $(x':y':z')$ upon it, and in particular it is the inflectional tangent to that determined by P itself. And there are no other points $(x':y':z')$ which have this property. For they must lie on the Hessian of (7), which is a sextic in $x'y'z'$ when the latter are put for $\lambda\mu\nu$, and hence must be the same sextic.

If the cubic (7) has a double point, it must be on the sextic. Thus the envelope contains the points $(x':y':z')$ whose corresponding cubics (7) have double points. For the latter count for two intersections with the sextic in each case, and there will be only three lines of the system through $(x':y':z')$. And the envelope in general contains no other points. For the cubic (7) determined by an arbitrary $x'y'z'$ is tangent at its four intersections with the sextic that depend on $x'y'z'$ to the corresponding line of the system, and in general there are no points on the sextic at which this line coincides with the sextic. The condition that (7) have a double point is the vanishing of the eliminant of the first partial derivatives of (7) and of its Hessian. This is of degree twelve in xyz.

7. *The introduction of relations among the seven original points leads to transformations of lower order, each of which has an invariant curve.*

Something should be said of the special cases produced by introducing relations among the seven original points. It was noted in section 3 that the coördinates of 9, $\lambda'\mu'\nu'$, as given by (2) contain the common factor $(\lambda'\cdot\mu_2\cdot\nu_3)$ which vanishes when 1 2 3 are collinear. This does not mean that 9 is indeterminate if three of the seven points are on a line. The difficulty is obviated by numbering them suitably. It is very easy to deal with the special cases if we recollect in connection with the elementary properties of the cubic the geometric character of the Cs and Ks in the equations of transformation (2) or (2′).

If three points are collinear they may be numbered 2 3 7. Then all the steps taken in deriving (2) may be retraced and no equation becomes illusory. But $K_1\ K_2\ K_3$ all contain a linear factor corresponding to the line 237. This linear factor is therefore common to $\lambda'\ \mu'\ \nu'$ and appears in the invariant sextic. We have, therefore, a transformation of degree seven and an invariant quintic curve, whose properties are immediately deduced by regarding it as part of the sextic.

Thus the quintic will have double points at 1 4 5 6, and will pass through 2 3 7. Since 2 3 7 are collinear, the cubic through the seven fixed points that has a double point at 2 must consist of the line 2 3 7 and the conic 1 4 5 6 2. Hence this conic and the quintic have the same tangent at 2. Corresponding relations hold at 3 and 7. Proceeding thus, the quintic is seen to fulfill fifty-three conditions. Removing the common factor, λ' μ' ν' represent curves of order seven, each of which has a triple point at 1 4 5 6 and a double point at 2 7 3. This gives forty-eight common intersections and imposes thirty-three conditions, thus satisfying the requirements for a birational transformation of degree seven.

Other cases are dealt with in similar fashion. If four of the seven points are collinear, 9 is necessarily indeterminate. If six are on a conic they may be numbered 4 5 6 1 2 7. Then C_1 C_2 C_3 all contain a corresponding quadratic factor, which appears to the first degree in the sextic, and to the second degree in λ' μ' ν'. We have thus a quartic transformation with an invariant quartic curve. This quartic has a double point at 3, and its tangents at the other six fixed points are the lines joining them respectively to 3. Hence the class is at least ten; and since 3 is known to be a double point, it is exactly ten, and the deficiency is two. The same is true if 3 should be a cusp; for from a cusp three less tangents can be drawn than from an arbitrary point. This indicates that in the general case the only singularities of the non-degenerate sextic are those at the fixed points, and that its deficiency is three. If the six points lie on a proper conic, the quartic fulfills forty-seven conditions, relatively the largest number satisfied by any of the invariant curves connected with this transformation. If the six points lie on two lines, three points on each, the quartic satisfies thirty-five conditions. The coördinates of 9 represent quartic curves which have a common triple point at 3 and single intersections common at each of the other six points—a well known type of birational transformation of degree four. Since 4 5 6 1 2 7 are on a conic, 9 must lie on 3 8. This line meets the quartic twice at 3, and 9 is the harmonic conjugate of 8 with respect to the other two intersections of 3 8 with the quartic. If 8 is at 3, 3 8 is any line through 3, and thus to 3 will correspond a curve which is in fact the polar cubic of 3 with respect to the quartic, since any line through a double point of a quartic is divided harmonically by the quartic and the polar cubic. But this polar cubic has a double point at 3 and passes through the other six fixed points, since the tangents to the quartic at those points are the lines joining them to 3. To each of these six points corresponds the line joining it with 3.

If 4 5 6 1 2 7 are on a conic and 3 is on 2 7, the transformation is of degree three, and the invariant curve is a cubic of deficiency one. For it passes through 3 1 4 5 6, and its tangents at the last four points are 3 1, 3 4, 3 5, 36. The cubic fulfills twenty-nine conditions if the fixed conic does not degenerate, and twenty if it does. λ' μ' ν' represent cubic curves having a common double point at 3 and single intersections common at 1 4 5 6. 9 is again the harmonic conjugate of 8 with respect to the other two intersections of 3 8 with the cubic. Thus to 3 corresponds its polar conic with respect to the cubic. To 1 4 5 6 correspond the lines which join them respectively to 3.

The case when 4 5 6 1 2 7 are on a conic and 3 is the intersection of 2 7 and 1 6, is particularly interesting. The transformation is quadratic, and the invariant curve is a conic which, if the fixed conic does not degenerate, satisfies sixteen conditions. For it is tangent to 3 4 at 4, and to 3 5 at 5, and passes through the intersections of 1 5 with 4 6, of 75 with 2 4, of 1 4 with 5 6, of 7 4 with 5 2, of the conic 3 2 4 5 6 with 1 7, of 3 2 1 4 5 with 6 7, of 3 4 5 1 7 with 6 2, and of 3 4 5 6 7 with 1 2. 9 is again the fourth harmonic of 8 with respect to the intersections of 3 8 with the invariant conic, that is, the intersection of 3 8 with the polar of 8. The canonical form of this transformation is $x'=yz$, $y'=xy$, $z'=zx$. Inversion with respect to the unit circle is a special case. All these curves exist apart from the transformation that revealed them. The arbitrary numbering in each case simply serves to adapt it to the original procedure. Thus in the last case we may say that if six points are taken on a proper conic, and if any vertex of the complete quadrangle of any four is taken as a seventh point, there is a conic which has the properties just described. Finally if 4 5 6 1 2 7 are on a conic, and if 2 7, 1 6, and 4 5 are concurrent at 3, the transformation is a collineation. The invariant line is the polar of 3 with respect to the fixed conic, and 9 is the harmonic conjugate of 8 with respect to 3 and the intersection of 3 8 with the invariant line. 3 is a self-corresponding point.

Whenever six of the fixed points are on a conic, 9 must lie on the line joining 8 and the remaining point, unless 8 is on the fixed conic, in which case 9 is any point on that conic. It was observed in the case of the quartic, the cubic, and the quadratic transformations that the line connecting 8 with this remaining point had two intersections with the invariant curve that depended on 8. Hence in those cases the envelope considered in section 6 is indeed of class four; but it is simply the fixed conic and the remaining point (a curve of class one) counted twice. The cubics C_1, C_2, C_3 are not linearly independent when 4 5 6 1 2 7 are on a conic, but the results just stated may be verified in certain simplified cases by use of l_p r l_s as in section 6, paragraph 1.

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